Deflection formulae for beams and properties for equivalent beam models for parallel chord trusses and vierendeel frames

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1. General formula

$$\Delta = \Delta_b + \Delta_s = \frac{C_b W L^3}{EI} + \frac{C_s W L}{K_s}$$

Beam deflection: Δ

Deflection due to bending deformation: Δ_b

Deflection due to shear deformation: Δ_s

C_b and C_s from Table 3

Expressions for I see Table 1

Expressions for K_s see Table 1

W - total load
L - span
EI - bending stiffness parameter
K_s - shear stiffness

2. Beam

Table 1 Expressions for *I* and K_s

Structure	***I***	K_s
Beam	*I*	$A_s G$ - Table 2
Parallel chord truss	I_g	K_{st} - Eq (A)
Vierendeel frame	I_g	K_{sv} - Eq(B)

$$I_g = \frac{A_c b^2}{2}$$

Table 2 Values for shear area A_s

Section	A_s
Rectangle b x d	$\frac{5}{6} bd$
I section bent about major axis	*Area of web*
I section bent about minor axis	5/6 *Area of flanges*

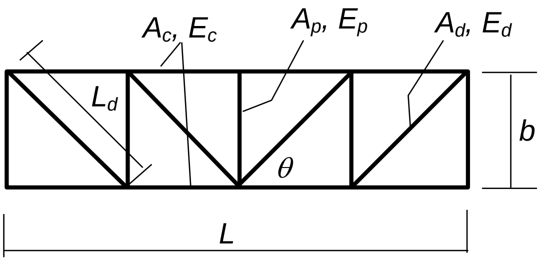
$$G = \frac{E}{2(1+\nu)}$$

3. Parallel chord truss

$$K_{st} = \frac{1}{\dfrac{1}{f E_d A_d \sin^2\theta \cos\theta} + \dfrac{1}{E_p A_p \operatorname{cotan}\theta}} \quad \text{(A)}$$

f = 1.0 for singly braced truss
= 2.0 with compressive cross bracing
= 0.5 for K bracing

With tensile only cross bracing treat as singly braced
With compressive cross bracing ignore flexibility of posts

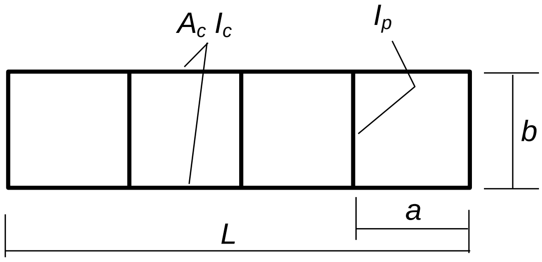


Parameters for parallel chord truss

4. Vierendeel frame

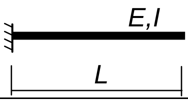
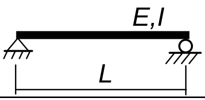
$$K_{sv} = \frac{24EI_c}{a^2[1+2\psi]} \qquad \text{(B)}$$

$$\psi = \frac{I_c / a}{I_p / b}$$



Parameters for vierendeel frame

Table 3 Beam deflection coefficients

Structure	**Load**	**C_b bending**	**C_s shear**
Cantilever 	Point tip	1/3	1.0
	UD	1/8	1/2
Simply supported 	Point central	1/48	1/4
	UD	5/384	1/8

5. Derivation of K_{st}

From the Bar Element Document, Equation (21) is:

$$\Delta = \delta_1 + \delta_2 = \frac{WL_d}{(EA)_d \cos^2\theta} + \frac{WL_p}{(EA)_p}$$

Governing differential equation for shear deformation:

$S = K_s \, dv/dx$

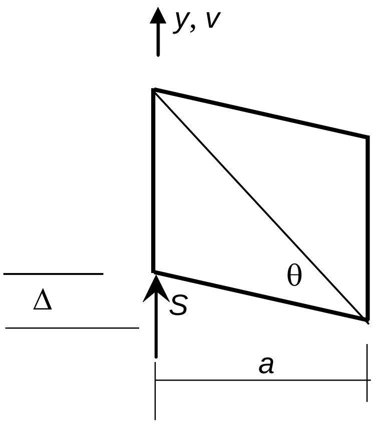
i.e. $K_s = \dfrac{S}{dv/dx}$

v is the displacement in the y direction

From the diagram: $dv/dx = \Delta/a$

Substituting this and $S = W$ into (21):

$$\frac{dv}{dx} = \frac{S L_d}{a\, E_d A_d \sin^2\theta} + \frac{S b}{a E_p A_p}$$



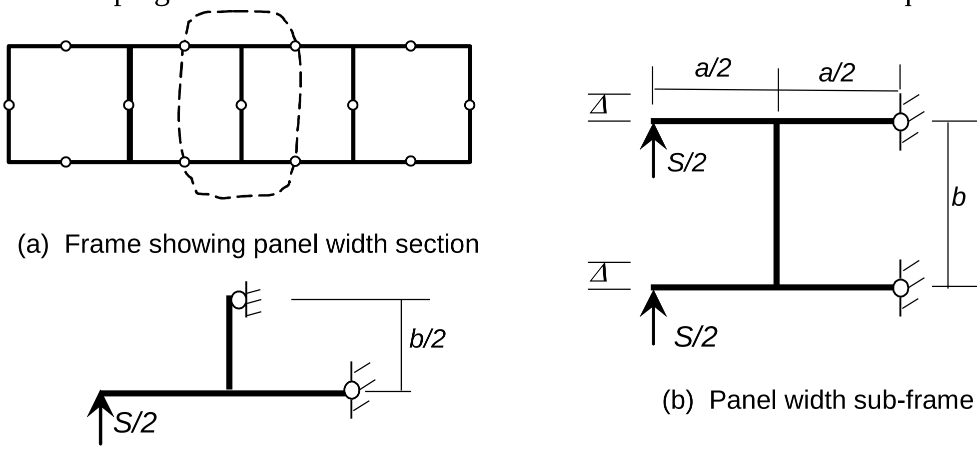
Note that $\sin^2\theta$ is used because the θ for the frame is (90 - θ) for Equation (21)

Using $a/L_d = \cos\theta$ and $a/b = \cot\theta$:

$$K_s = \frac{S}{dv/dx} = \frac{1}{\dfrac{1}{E_d A_d \sin^2\theta \cos\theta} + \dfrac{1}{E_p A_p \cot\theta}}$$

6. Derivation of K_{sv}

Figure 1(a) shows a vierendeel frame with points of contraflexure at mid-length of all members. Such positions for the points of contraflexure is the fundamental assumption in developing the shear mode deformation of a vierendeel frame as an equivalent beam.



(a) Frame showing panel width section

(b) Panel width sub-frame

(c) Symmetrical half of sub-frame

Figure 1 Vierendeel frame

Also shown on Figure 1(a) is a section of the frame bounded by points of contraflexure. This is extracted to Figure 1(b) where the shear at the points of contraflexure $S/2$. S is the total shear at the section and half is taken by each chord (assuming them to have the same I value). The final trick is to work on a symmetrical half of this sub-frame as in Figure 1(c). The deflection under the S/2 load of the frame of Figure 1(c) is calculated (using the principle of virtual work) to be:

$$\Delta = \frac{S a^3}{24 E I_c}[1 + 2\psi] \qquad \text{where } \psi = \frac{I_c / a}{I_p / b}$$

hence $dv/dx = \Delta/a$ and $K_{sv} = S/(dv/dx)$

$$\text{hence} \quad K_{sv} = \frac{24 E I_c}{a^2 [1 + 2\psi]}$$

Note $G = E/(2(1+\nu))$ where ν is Poisson's Ratio

7. Constitutive relationships for bending and shear:

Bending

$M = EI\, d^2v/dx^2$ i.e $d^2v/dx^2 = M/EI$

Shear

$S = K_s$ i.e. $dv/dx = S/K_s$

where K_s is the shear stiffness.

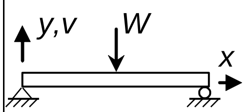
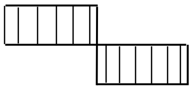
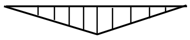
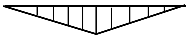
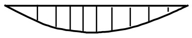
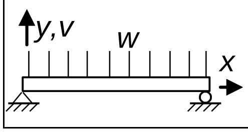
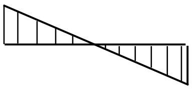
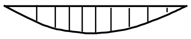
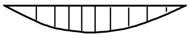
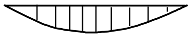
For a beam $K_s = A_s G$ where A_s is the shear area and G is the shear modulus.

Table 3 shows shapes of shear force and bending moment diagrams and corresponding displacement diagrams.

For bending, the basic relationship needs to be integrated twice to get the displacement. therefore the function for the displaced shape is two orders higher than that for the bending moment e.g. from Table 3 with UD load, the bending moment is parabolic whereas the displacement is quartic (fourth order).

For shear, the basic relationship needs to be integrated once to get the displacement and therefore, for example, with a point load, the shear is constant and the displacement is linear.

Table 4 Diagram shapes

	Shear		**Bending**	
Loading	**Shear force**	**Displacement**	**Bending Mom**	**Displacement**
Point 	Constant 	linear 	linear 	cubic 
UD 	linear 	parabolic 	parabolic 	quartic 

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