Force, Equilibrium, Stress

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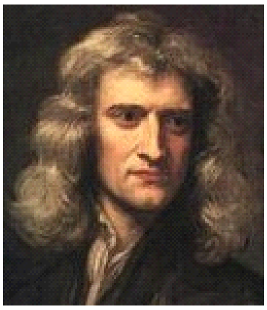
1 Newton's Laws of Motion

We all understand the effect of forces; all physical objects are continuously subject to the action of forces But the physics of force tends not to be clearly explained in textbooks.

The normal starting point is Newton's Laws of Motion. A direct translation of these from the Latin of Newton's *Principia* is[1]:

1. Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.
2. The rate of change of momentum of a body is proportional to the resultant force acting on the body and in the same direction.
3. All forces occur in pairs and these two forces are equal in magnitude and opposite in direction.



Isaac Newton

Law 1 states that a force is needed to change the motion of a body.
Law 2 states a proportional relationship between a force and the rate of change of momentum.
Law 3 can be interpreted as a statement of the principle of equilibrium.

Laws 1 and 2 relate to bodies in motion and address only one type of force i.e. that due to change in momentum. There are several other types of force as discussed in Section ??

2 Equilibrium

2.1 Basics

Equilibrium is one of the most important principle in structural mechanics so when we talk about it, it is important to know what is meant. Unfortunately the use of the term 'equilibrium' is by no means precise.

A mathematical statement of Newton's Second Law of Motion is that for any body:

$$d(MV)/dt = \Sigma F_i \qquad (1)$$

where M is the mass of the body, V is the velocity and ΣF_i is the sum of the forces on the body. V and F_i are vectors which act along a common line. MV is the momentum of the body.

Assuming the mass to be constant, Equation (1) is more commonly written as:

$$M dV/dt = Ma = \Sigma F_i \qquad (2)$$

where $a = dV/dt$ is the acceleration of the body,

The physicists' approach is to say that the equilibrium condition is when there is no change in momentum such that from Equation (1) :

$$\Sigma F_i = 0.0 \qquad (3)$$

Equation (3) is a statement of *static equilibrium*.

Equation (2) suggests that Ma is not a force. Physicists treat it as a 'fictitious force' (apparently this distinction is important when taking account of relativity). However the engineers' approach is to treat Ma as an *inertia force* and write expression (2) as:

$$\Sigma F_r = \Sigma F_a \quad (4)$$

where F_r is a restraining force pulling back on the body and F_a is an active force which pushes it forwards. The inertia force is a restraining force

For example the commonly used equation for the dynamic motion of a system consisting of a mass M restrained by a spring of stiffness K and also restrained by a viscous damper (i.e. with damping proportional to velocity) with damping constant C (Figure 1) is written as:

$$M\mathrm{d}^2u/\mathrm{d}x^2 + C\,\mathrm{d}u/\mathrm{d}x + Ku = P(t) \quad (5)$$

where u is the displacement in the x direction, $\mathrm{d}u/\mathrm{d}x$ is the velocity in the x direction, $\mathrm{d}^2u/\mathrm{d}x^2$ is the acceleration in the x direction and $P(t)$ is a 'forcing function' which is a time dependent active load on the mass. Typical forcing functions could be due to out of balance machinery forces. blast loading, etc.

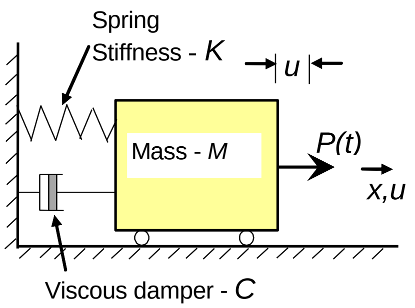


Figure 1 Mass and spring with viscous damping

Equation (5) is an example of Equation (4) and engineers call it an equilibrium equation. Physicists say this is wrong - the condition of equilibrium is when there is no acceleration i.e. is characterised by Equation (3).

As far as use is concerned Equations (2) and (4) are equivalent. In (2) all the terms except Ma are collected on one side as ΣF_i and in (4) all the restraining terms are on one side of the equation and the forcing terms are on the other.

So the nub of the problem of how to define 'equilibrium' lies in the distinction between Equations (2) and (4). In non-technical contexts use of 'equilibrium' is towards Equation (2). If one falls over and starts to accelerate towards the ground one says that one has 'lost equilibrium'.

2.2 Force as a vector

Force is a vector quantity which can be resolved into several equivalent forces as discussed in Section?? Instead of using Equation (4) where the forces are separated into resisting and exciting components, it is also common to write the equation of equilibrium as

$$\sum \overrightarrow{F_i} = 0.0 \quad (6)$$

i.e. as the vector sum of all the forces acting on the body. For example, using this approach Equation (5) would be written as: $P(t) + M\mathrm{d}^2u/\mathrm{d}x^2 + C\,\mathrm{d}u/\mathrm{d}x + Ku = 0.0$ and the values used for the restraining forces would be negative.

Validation for the equilibrium condition.

Using equation (4), equilibrium is a universal condition which holds in all situations. Even when you are out of balance Equation (4) is valid to a high degree of accuracy for velocities that are not close to the speed of light. Its validity depends on (a) whether all the relevant types of force have been included and (b) whether these terms have been accurately modelled For example, in modelling a real system using Equation (5):

- there will be friction forces which are not included in the equation
- there will be some degree of error in the measurement of mass, stiffness, damping coefficient and forcing function.

3 Types of force

3.1 Momentum force

The term *momentum force* is used here to denote the force needed to change the motion of a body according to Newton's second Law i.e.

$$\text{i.e.}\quad F_{momentum} = \frac{d(MV)}{dt} \tag{7}$$

3.2 Inertia force

An *inertia force* F_i occurs when a body of constant mass is accelerated i.e. when Equation (2) is relevant i.e.

$$F_{in} = M\,\mathrm{d}V/\mathrm{d}t = Ma \tag{8}$$

The inertia force is the property of mass which tends to resist any change in its motion. It is therefore a resistance force.

3.3 Rocket thrust

A force due to *rocket thrust* occurs when mass (in the form of gas) is ejected from a rocket. Using Equation (2) and assuming that the velocity of the ejected gas is constant - V_g:

$$F_t = V_g\,\mathrm{d}M_g/\mathrm{d}t \tag{9}$$

where: $\mathrm{d}M_g/\mathrm{d}t$ is the rate of flow of the mass of the gas at the rocket nozzle. This force acts on the rocket and is therefore an excitation force.

Example

Figure 2 shows a rocket in space. Firing of the rocket ejects gas at high velocity relative to the rocket causing the rocket velocity V_r to increase.
The rocket has mass M_r and the mass flow of the gas is $\mathrm{d}M_g/\mathrm{d}t = m$ kg/sec

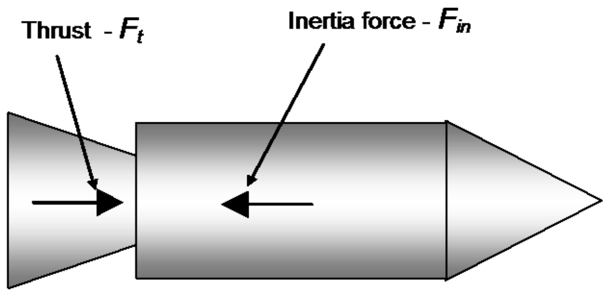


Figure 2 Fired Rocket

Acting on the rocket is a rocket thrust (Equation 9):

$$F_t = V_g\,\mathrm{d}M_g/\mathrm{d}t = V_g m \tag{10}$$

This is resisted by an inertia force:

$$F_{in} = M_r \, dV_r/dt = M_r a_r \qquad (11)$$

where a_r is the forward acceleration of the rocket.

The equilibrium equation based on (4) is therefore:

$$F_{in} = F_t \quad \text{i.e.} \quad M_r a_r = V_g m \qquad (12)$$

Validation

There are no drag forces to resist propulsion in space but the effect of loss of mass on the inertia force is neglected for Equation (10).

3.4 Spring force

Figure 3(a) shows a strut which is jacked against the side walls of an excavation. The jack force causes an axial force in the strut and a reaction at the walls . In this situation the value of the force is independent of acceleration or mass.

When a force is applied to a body there is always deformation. As the jack load is applied, the walls move apart and the length of the strut decreases. Strain energy is stored in the strut, in the jack and in the supports. The forces involved are static *spring forces* derived from strain energy.

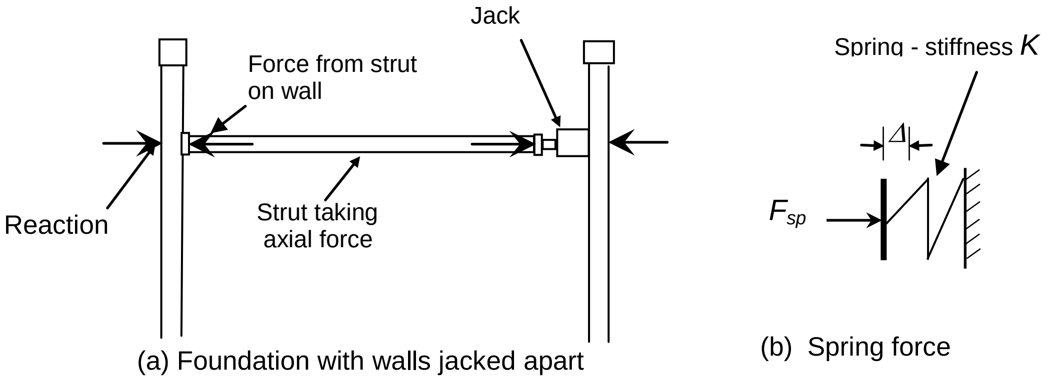


Figure 3 Spring forces

A relationship for a linear spring force (Figure 3(b)) - F_{sp} is:

$$F_{sp} = K \Delta \qquad (13)$$

where K is the spring stiffness and Δ is the movement of the spring.

3.5 Gravity force

Between any two masses there is a force field which can be treated a single attractive force - $F_{gravity}$ - acting in a line between the centres of gravity of the masses. Isaac Newton devised the Law of Gravity for this interaction:

$$F_{gravity} = (G M_1 M_2)/D^2 \qquad (14)$$

where G is the universal gravitational constant and D is the distance between the centres of gravity of the masses - Figure 4(a).

For an object at the surface of the earth of mass M_o, the gravity force will be:

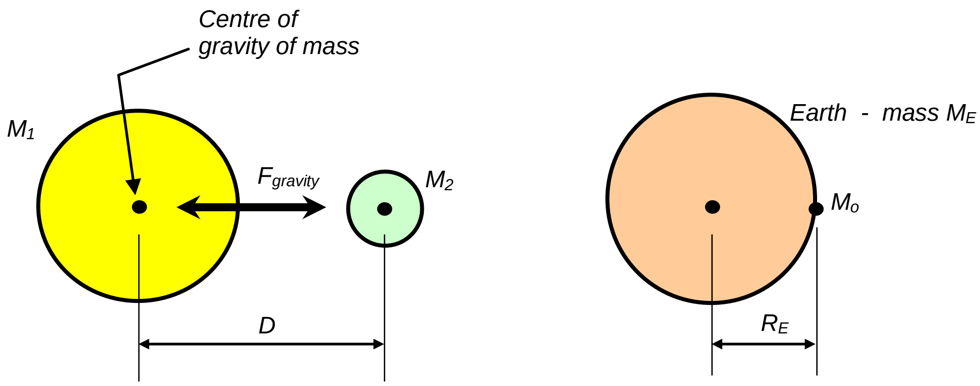
$$F_g = (G M_E M_o)/R_E^2 \qquad (15)$$

where F_g is the gravity force exerted by the earth on the mass M_o, M_E is the mass of the earth and R_E is the radius of the earth.

Substituting G = 6.67428 x 10^{11} m^3/kg/sec^2, M_E = 5.9736 x 10^{24} kg and R_E = 6378.135 km (at Equator) gives:

$$F_g = M_o \times 9.806 = M_o\, g \text{ kg m/sec}^2 \qquad (16)$$

where g = 9.806 is the gravity constant for a mass at the surface of the earth (the standard value is g = 9.8066 and g = 9.81 is normally used in engineering calculations.)



(a) 2 Masses distance D apart (b) Mass 2 at surface of the earth

Figure 4 Gravity forces

g is often referred to as the 'acceleration due to gravity' but this refers to the special case of an object in free fall at the surface of the earth with no other force actions on it (i.e. no frictional drag) - see section which follows.

Object in free fall

Figure 5 shows an object with mass M in free fall (no frictional drag) near the surface of the earth. An inertial force $F_{in} = Ma$ acts upwards and a gravity force $F_g = Mg$ acts downwards. The equation of equilibrium is then:

$$F_{in} = F_g \quad \text{i.e.} \quad Ma = Mg \qquad (17)$$

hence $a = g$

This is why g is called the 'acceleration due to gravity'. It is not a good name because Equation (17) is only valid if the mass is falling within a vacuum. It is easier to understand gravity force if g is treated as a constant that has the same dimensions as acceleration (LT^{-2}).

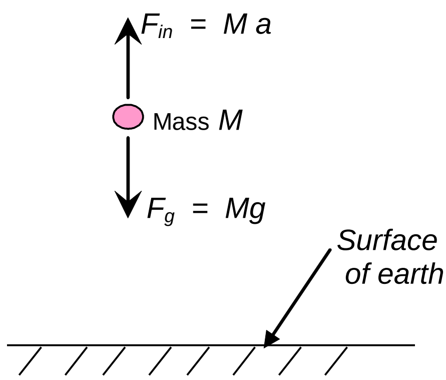


Figure 5 Mass in free fall

Static gravity force

Figure 6 shows a block of concrete supported on the ground. A gravity force:

$$F_g = M_b\, g \qquad (18)$$

where M_b is the mass of the block and g is the gravity constant. The gravity force acts downwards. This causes the support to

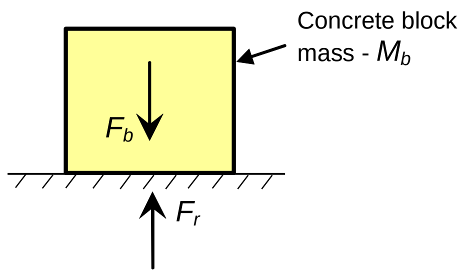


Figure 6 Supported Block

deform. A spring force reaction - F_r - acts upwards due to strain energy in the ground. The equilibrium condition is:

$$F_r = F_b \tag{19}$$

Validation

Three parameters contribute to the value of g using Equation (15) G, M_E and R_E. Of these the radius of the earth R_E has the greatest uncertainty. The earth is not exactly round - the radius is greater at the equator than at the poles and, of course, the radius depends on altitude (i.e. the height above (or below) sea level. Table 1 gives examples of the effect of some assumptions in the calculation of g.

Table 1 Error in values of g

Assumption	**Error relative to equator value (%)**
Use 9.80 rather than 9.81	0.1
R_E at poles (= 6356.75 km) rather than at equator	0.33
R_E at top of Mount Everest	0.1

3.6 Other types of force

Other types of force include friction force, damping force, centrifugal force. etc.

The standard for defining force is its inertial form, force = mass x acceleration with units MLT^{-2}. It would be possible to use a definition based on other force types, for example based on the deflection of a standard spring. Force would then have units of Length. It was common in the UK in the past to use a *pound force* unit being the gravity force exerted by a mass of one pound at the surface of the earth. Using this definition force has the units of mass.

4 Force Units

The standard for defining force is its inertial form, force = mass x acceleration with units - MLT^{-2}. The standard unit is the Newton which is the force required to cause a mass of 1 kg to accelerate by 1.0 m/sec^2. The SI symbol for Newtons is ‘N’.

The standard for defining force could be universal gravity i.e. a unit of force - a ‘U’ - would be the gravity force exerted between two masses each of 1.0 kg at a distance apart of 1.0 m. The Universal Gravity relationship would be:

$$F_{gravity} = (M_1 M_2)/D^2$$

i.e. a gravitational constant would not be needed and the units of force would be M^2/L^2.

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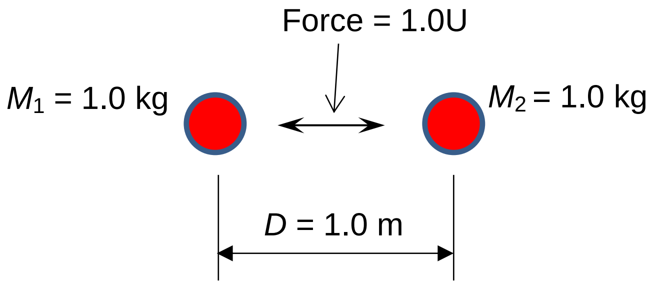


5 Free body diagrams and internal force actions

5.1 Definitions

A *free body* is a system or part of a system which is considered to be 'cut' (or separated) from its surroundings. At the cuts, force actions are applied to represent those which are present in the system. A *free body diagram* is a diagram which shows all the force actions on a free body.

The principle of equilibrium applies to a body and to any part of the body. Therefore the principle can be applied to a free body. The use of free body diagrams for considering equilibrium is a fundamental strategy in the use of structural mechanics.

Structures are considered to have *applied loads* (external actions) and *internal actions*. Typical applied loads are: gravity forces, force due to wind, force due to explosions (blast loading), etc. Internal force actions are the resultants of stresses within the system. They are identified when free body diagrams are created.

However the distinction between applied and internal force actions is not precise. The loads applied to a free body may be due to internal actions from the surrounding structure as discussed in Section It is best to think of internal force actions as a pair of equal and opposite action at a cut to create a free body - see Section 5.2

5.2 Example of free body diagrams

Figure 7(a) shows a person standing on bathroom scales. Figure 7(b) shows a model of the situation where the scales are represented as a spring the lower end of which is restrained from moving vertically. The person has mass 70 kg and therefore imposes a downward applied load of $F_g = Mg = 70 \times 9.81 = 687$ N (Equation 16).

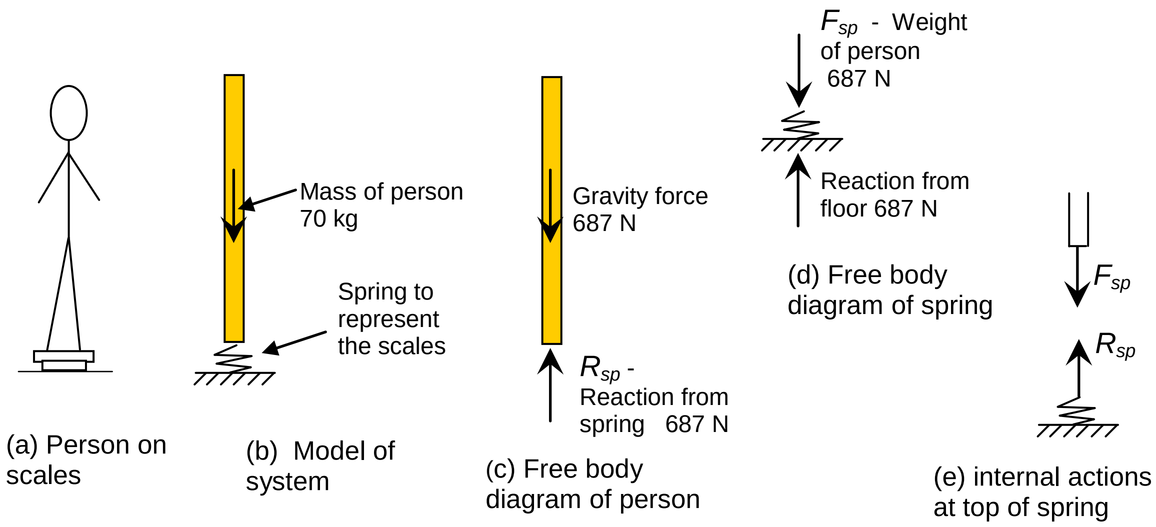


Figure 7 Free body diagrams for person on scales

Figure 7(c) is a free body diagram of the person. A gravity force of 687 N which represents the weight of the person acts downwards and for equilibrium there is an upward reaction R_{sp} from the spring.

Figure 7(d) shows a free body diagram for the spring. The load from the person acts downwards on the top of the spring and there is an upward reaction from the floor at the base of the spring.

Figure 7(e) shows the internal force actions between the top of the spring and the feet of the person. A pair of equal and opposite vertical forces acts. The upward value - R_{sp} - is the action of the spring on the feet of the person and may be considered to be an applied load when analysing the effect of forces on the person (e.g. when using the free body diagram of Figure 7(c)) .

The downward value - F_{sp} - is the action of the feet of the person on the spring and may be considered to be an applied load when analysing the spring on its own (e.g. when using the free body diagram of Figure 7(d)) .

Thus whether a force action is considered to be external or internal depends on the context.

5.3 Process for creating a free body diagram

The following actions should be taken:

1. Extract a 'free body' part of the structure for which equilibrium is to be applied.
2. At the 'cuts' which were made to create the free body apply force actions which exist there in the real system
3. Draw the free body diagram which shows the applied loading and the force actions at the cuts i.e. all the force actions which act on the body.

6 Applying the principle of equilibrium along a line

Collinear forces act along a common line. Figure 8 shows a body with three collinear forces acting on it.

Two approaches to writing the equilibrium equation for such a situation are:

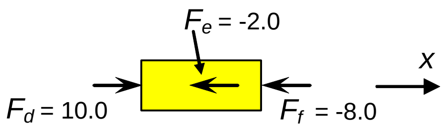


Figure 8 Forces along a line

1. *Set the vector sum of the forces to zero* .
   For the set of forces of Figure 8 the condition of equilibrium is:

$$F_d + F_e + F_f = 0.0 \qquad (20)$$

i.e. $10.0 + -2.0 + -8.0 = 0.0$

If the line is considered to be in the x direction then the general statement of the equilibrium condition along the line is:

$$\Sigma F_{xi} = 0.0 \qquad (21)$$

Where F_{xi} is a force in the x direction

2. *Sum of forces in one direction = Sum of forces in opposite direction.* For the forces on the body of Figure 8 this would result in the expression:

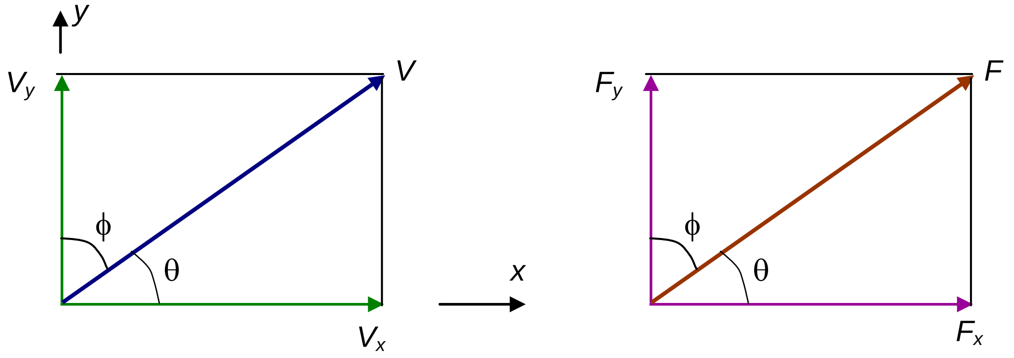
$$F_d + F_f = F_e \qquad (22)$$

i.e. $10.0 = 2.0 + 8.0$

7 Forces in a plane

7.1 Resolution of forces

Figure



(a) Vector components (b) Resolution of forces

Figure 9 Transformation of vectors

Figure 9(a) shows a vector V with components V_x and V_y in the x and y directions respectively. The co-ordinate axes are cartesian (i.e at right angles to each other) and the relationships between V and its components is:

$$V_x = V\cos\theta \quad \text{and } V_y = V\cos\phi \tag{23}$$

$\cos\theta$ and $\cos\phi$ are the direction cosines of V with respect to the x and y axes.

Figure 9(b) shows a force F with components in the x and y directions. The relationships between F and its components are:

$$F_x = F\cos\theta \quad \text{and} \quad F_y = F\cos\phi$$

or in matrix notation:

$$\begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \begin{bmatrix} \cos\theta \\ \cos\phi \end{bmatrix} F \tag{24}$$

That forces can be represented by components in the same way as a vector is a matter of observation. The transformation of Equations (24) is known as *resolution of forces.*

The use of Equations (24) is a main strategy for defining equilibrium conditions in two dimensions. Since $\cos(90 - \theta) = \sin\theta$ instead of using $\cos\phi$ it is common to use $\sin\theta$, i.e. (24) is normally written as:

$$F_x = F\cos \quad \text{and} \quad F_y = F\sin\theta \tag{25}$$

A result of treating force as a vector is that if it is not parallel to a co-ordinate axis then it can be replaced by components in these directions. This is used, for example, when dealing with diagonally braced trusses.

7.2 Equilibrium of forces in a plane

Figure 10(a) shows a body with three applied forces:

- 8.0 is in the negative x direction
- 6.0 in the negative y direction
- 10.0 at an angle θ = 36.87^{o} to the x axes. (This is the angle for a 3-4-5 triangle such that cosθ = 0.8 and sinθ = 0.6)

The process for applying equilibrium to this system is:

1. Resolve all forces into the x and y directions using Equations (24). The 10.0 force has components (Figure 10(b)):

   x direction 10.0 cosθ = 10.0*0.8 = 8.0

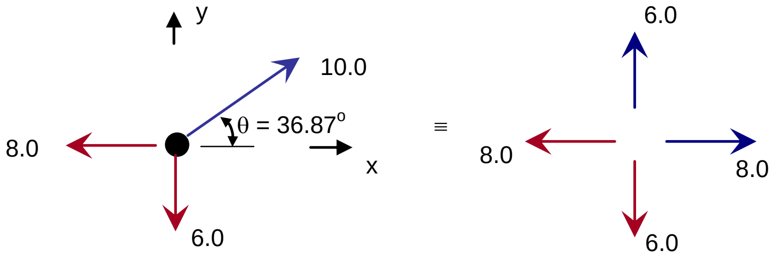
   y direction 10.0 sinθ = 10.0*0.6 = 6.0

2. Apply the condition of equilibrium to the forces in the x and y directions separately

   i.e. $\Sigma F_{xi} = 0.0 = -8.0 + 8.0$

   $\Sigma F_{yi} = 0.0 = -6.0 + 6.0$

This demonstrates that the equilibrium condition is satisfied for the system of Figure 10.



(a) Force system at a point   (b) Diagonal force resolved

Figure 10 Equilibrium of forces at a point in a plane

7.3 Resultants and equilibrants

Figure 11(a) shows a pair of forces represented as vectors F_a and F_b. Vector theory shows that the diagonal of the parallelogram OC (which represents the force F) formed as in Figure 11(a) is the equivalent of the two vectors F_a and F_b. (This is a more general case of resolution of forces than that of Figure 9.

The diagonal force F is the *resultant* of F_a and F_b.

The *resultant* of a set of forces is the single force which has the same effect as the set.

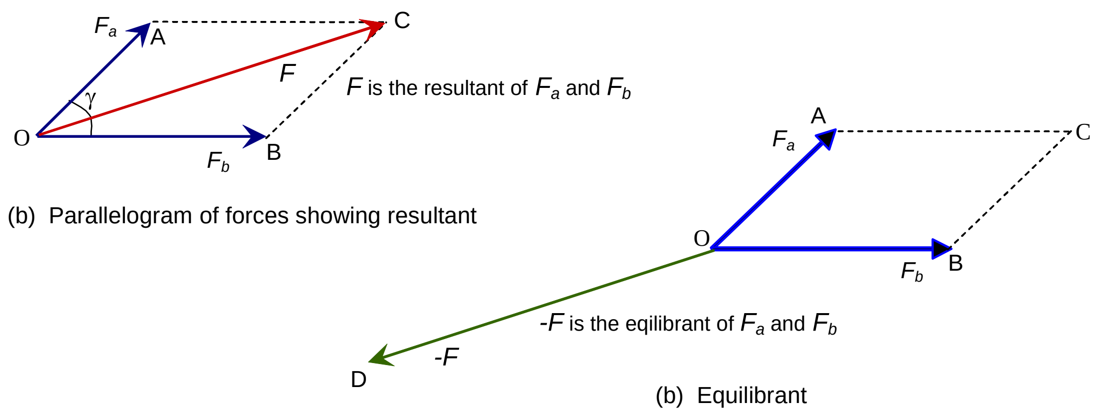


Figure 11 Resultant and equilibrant

Figure 11(b) is the same diagram as 11(a) but with the line OD drawn to the same length but in the opposite direction from OC. (OD looks longer than OC but this is an optical illusion). The line OD representing the force $-F$ in scale and direction, represents the *equilibrant* of F_a and F_b.

The *equilibrant* of a set of forces is the single force which balances the set. It is has the same magnitude and acts along the same line as the resultant but in the opposite direction.

8 Equilibrium in 3 dimensions

8.1 Resolution of forces in 3 dimensions

The component of a force F resolved into Cartesian axes is given by $F\cos\beta$ where 'β' is the angle between the direction of the force and the direction of the axis.

This holds in three dimensions (Figure 12)):

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{Bmatrix} \cos\beta_x \\ \cos\beta_y \\ \cos\beta_z \end{Bmatrix} F \qquad (26)$$

$\cos\beta_i$ is the direction cosine of F_i for resolution of a force in 3 dimensions. β_x, for example, is the true angle angle between the line of F and the x axis.

Calculating the β angles

The β angles can be calculated as follows (Figure 12): Establish a point A which is distance L from the origin of the coordinates - O. The distance OA should represent the force F in magnitude and direction. From A draw a line perpendicular to the xz plane to intersect this plane at B. From B draw lines parallel to the x and z axes to intersect these axes at points D and C respectively.

The direction cosines are then given by:

$\cos\beta_x$ = OC/L

$\cos\beta_z$ = OD/L/L

$\cos\beta_y$ = BA/L

where L^2 = $OC^2 + OD^2 + BA^2$

The projection from A can alternatively be made to planes xy or yz.

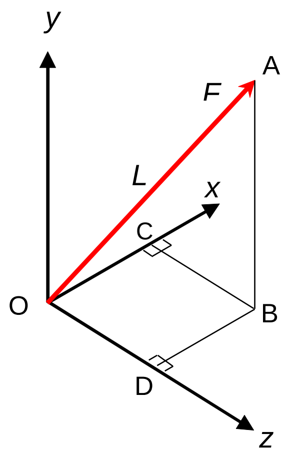


Figure 12 Calculation of direction cosines

8.2 Applying equilibrium in 3 dimensions

The process for applying equilibrium in 3 dimensions at point in space is the same as for 2 dimensions but with an extra axis:

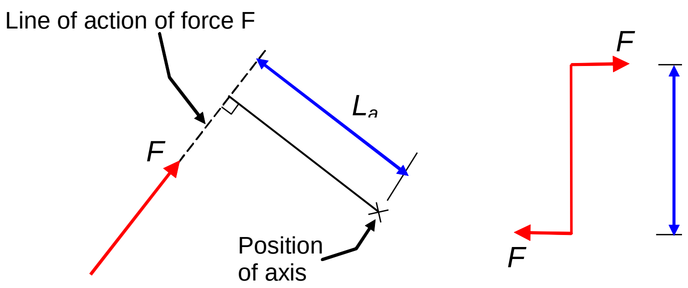
1. Resolve all forces which are not parallel to a coordinate axis into the coordinate axes using (26).
2. Apply the condition of equilibrium (Equation 21) in the 3 coordinate directions separately.

9 Moments

9.1 Definitions

The term *force* normally denotes a *direct force* which acts along a line. The forces considered in Sections 1 to 8 are direct forces.

A *moment* is a turning effect caused by a direct force acting about an axis.



(a) Lever arm L_a (b) Couple

Figure 13 Definition of a moment

The value of a moment is:

$$M = F \times L_a \qquad (27)$$

where:

- F is a direct force
- L_a is the *lever arm* - the perpendicular distance from the axis about which the moment is to be calculated to the line of action of the force - Figure 13(a)

A *couple* is a pair of equal and opposite forces acting over a lever arm distance L_a - Figure 13(b). The value of the moment - M - is the same in both cases as defined by Equation (27).

Note that there is no resultant direct force with the couple. If a pure moment is applied to a body it must be in the form of a couple.

A moment has dimensions typically Newton metres - N m.

A *force action* is either a direct force or a moment. The term 'force actions' denotes a set of direct forces and/or moments.

9.2 Calculating the lever arm for a moment

A process to calculate the lever arm for a force in a plane about an axis which is at right angle to the plane is:

1. In the plane, draw a line which represents the vector of force. Extend it as far as is necessary for Step 3 - Figure 13(a)
2. Identify the point on the plane to represent the axis about which the moment is to be calculated.
3. From the point draw a line at right angles to the line of action the force
4. The distance from the point to the line of action is the lever arm

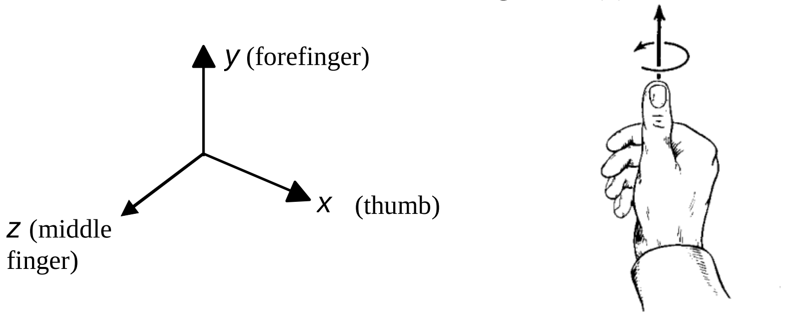
9.3 Force as a line vector

In general a vector has magnitude, direction and position but for equilibrium calculations a force are treated as line vector. For a *line vector* the position of the force along which it acts is not important to the process. This is why the line of action of the force can be extended to calculate the value of the lever arm

Treating force as a line vector also allows forces along a line to be summed for equilibrium even if they do not all act at the same point on the line - see for example Figure 8.

9.4 Sign conventions for moments

Coordinate axes The standard convention for Cartesian coordinate axes (i.e. axes at right angles to each other) is to use the *right hand rule.* Hold the thumb and first two fingers of the right hand at right angles to each other. Point the thumb in the direction of the x axis, the forefinger in the direction of the y axis. The middle finger now points in the direction of the z axis - Figure 14(a)



(a) Right hand rule for orientation of axes

(b) Right hand screw rule for moments

Figure 14 Sign Conventions

Moments The standard convention for moments is the *right hand screw rule*. Point the thumb of the right hand in the positive direction of an axis with the hand partially closed - 14(b). The fingers point in the positive direction of moment. Another way to define the same convention is to say that positive moment is clockwise looking down the axis in the positive direction.

9.5 Moment equilibrium

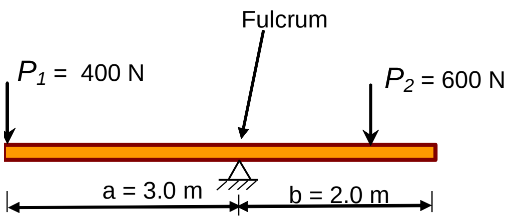


Figure 15 Equilibrium of a see-saw

Figure 15 shows a see-saw. If a child sits on one end and an adult on the other then the adult end will go down. To balance the system the adult moves towards the fulcrum (i.e the support for the see-saw about which it rotates) as in Figure 15.
If the gravity force of the child is 400N (corresponding to a weight of about 40 kg) and the corresponding force of the adult is 600 N then the moment of the child about the fulcrum is:

$$M_1 = P_1 a = 400*3.0 = 1200 \text{ Nm}$$

where a is the lever arm of P_1 about the fulcrum.

The moment of the adult about the fulcrum is:

$$M_2 = P_2 b = 600*2.0 = 1200 \text{ Nm}$$

where b is the lever arm for P_2 about the fulcrum

For equilibrium, these are equal and opposite. The see-saw is in balance.

Using the sign convention positive clockwise, the formal condition of moment equilibrium about the fulcrum is:

$M_1 + M_2 = 0.0$ i.e. $-1200 + 1200 = 0.0$

The general statement of moment equilibrium is:

The sum of the moments of all forces about any axis is zero

i.e. $\Sigma M_i = 0.0$ (28)

where M_i is the moment of force i about the axis. For example M_x is the moment of a force about the x axis. This naming convention is not universal. For example in traditional plate bending theory M_x is the moment acting on the x plane[2].

Note that the condition is valid for any axis of the system being considered but applying the condition again to the same system to a second axis which is parallel to the first provides no additional information.

9.6 Resolution of moments

Moments can be resolved into component directions as for direct forces. Figure 16 shows the axis of a force F which acts at right angles to the plane of the diagram. It acts away from the viewer of the diagram. Reference Cartesian axes x y are shown with a third axis n in the xy plane at an angle θ to the x axis.

F has lever arms:

- L about the n axis.
- L_x about the y axis
- L_y about the x axis

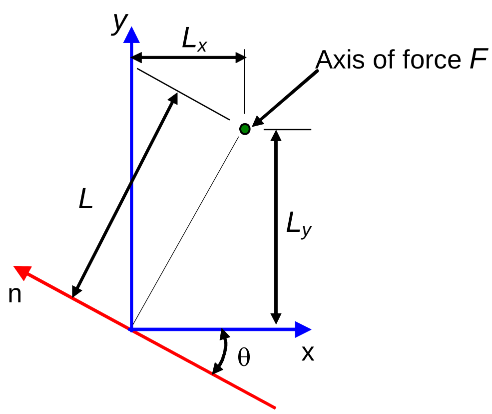


Figure 16 Resolution of a moment

Noting that: $L_y = L\cos\theta$ and $L_x = L\sin\theta$

and using the right hand screw rule (Section 9.4), the moments of F about the n, x and y axes are:

- $M_n = FL$ about the n axis
- $M_x = FL_y = FL\cos\theta = M_n\cos\theta$ about the x axis
- $M_y = -FL_x = -FL\sin\theta = -M_n\sin\theta$ about the y axis

Therefore the components of M_n are:

$$\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} FL \qquad (29)$$

The trick in transforming moments to a different set of axes is to draw a diagram like Figure 16 showing the moment as a force acting on the end of an arm and work out the lengths of the lever arm for moments about the relevant axes.

9.7 Equilibrium for moments which are not defined about the reference axes

The process for applying equilibrium for moments which are not defined about the reference axes is:

1. Resolve all moments so that they all defined about the reference axes using coordinate axes using Equation (29).
2. Apply the condition of moment equilibrium (Equation 28) for moments about the reference axes separately.

10 Particular forms of the condition for equilibrium

10.1 In-plane actions

In a plane (e.g the xy plane) there are three independent force actions - two direct forces F_x and F_y and a moment M_z - Figure 17(a). Therefore there are three independent conditions of equilibrium for a body in a (xy) plane, i.e.

$$\Sigma F_{xi} = 0.0, \qquad \Sigma F_{yi} = 0.0 \qquad \Sigma M_{zi} = 0.0 \tag{30}$$

where F_{xi} is a force in the x direction, F_{yi} is a force in the y direction and M_{zi} is moment about the z axis.

10.2 Out of plane actions

In a plate bending situation there are three independent force actions - a direct force F_z at right angles to the plane of the plate and two out of plane moments M_z and M_y - Figure 17(b) Therefore there are three independent conditions of equilibrium for a plate in bending:

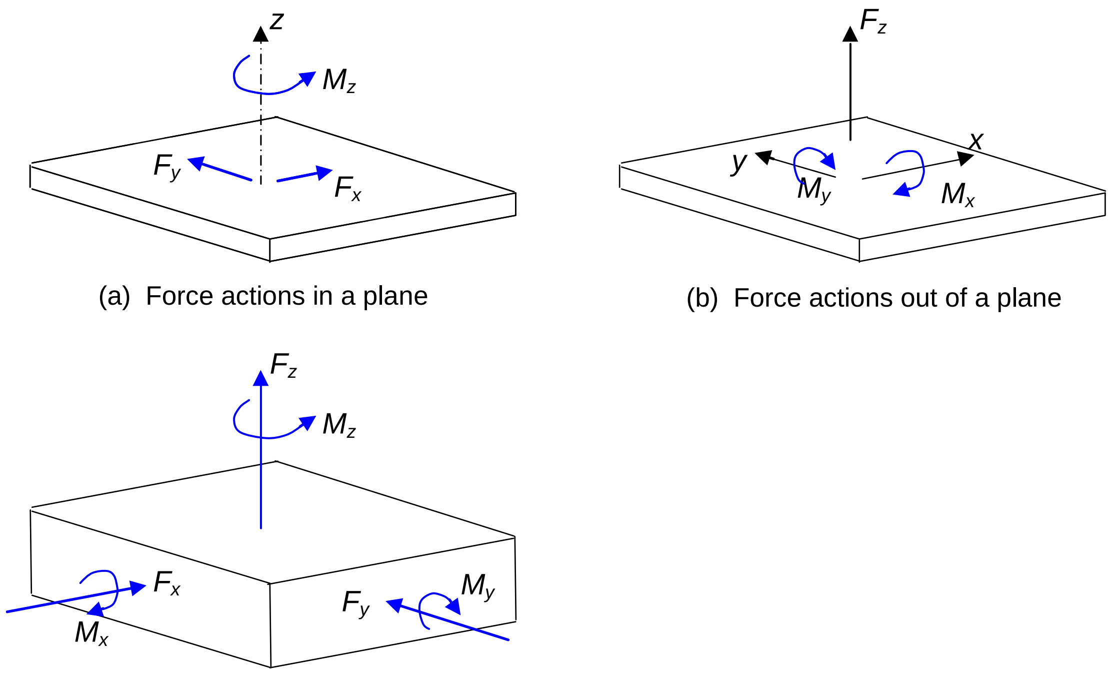
$$\Sigma F_{zi} = 0.0, \quad M_{xi} = 0.0, \quad \Sigma M_{yi} = 0.0 \tag{31}$$

where F_{zi} is a force in the z direction, M_{xi} is a moment about the x axis, and M_{yi} is moment about the y axis.

10.3 Actions in 3 dimensions

In three dimensions there are 6 independent force actions - three direct forces and three moments - Figure 17(c). Therefore there are six independent conditions of equilibrium for a body defined in 3 dimensions:

$$\begin{aligned} &\Sigma F_{xi} = 0.0, \quad \Sigma F_{yi} = 0.0, \quad \Sigma F_{zi} = 0.0 \\ &\Sigma M_{xi} = 0.0 \quad \Sigma M_{yi} = 0.0, \Sigma M_{zi} = 0.0 \end{aligned} \tag{32}$$



(a) Force actions in a plane

(b) Force actions out of a plane

(c) Force actions in a 3 dimensions

Figure 17 Force action sets

11 Stress

11.1 Basic Definitions

Stress is force intensity - force/unit area on a surface

A *normal force* - N - acts at right angles to the surface on which it is considered to act - Figure 18(a).
A *shear force* - N - acts parallel to the surface on which it is considered to act - Figure 18(b).
The corresponding stresses are:

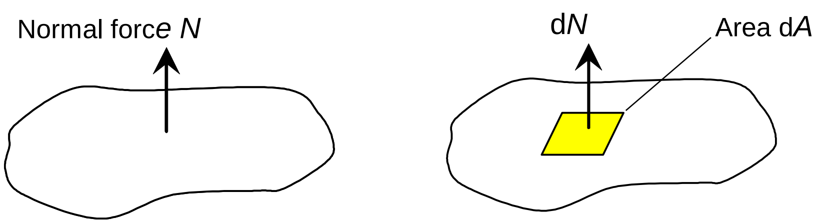
- *Normal stress* (also called *direct stress*) resulting from normal force - often denoted by the symbol σ - Figure 18(a). Direct stress is normal force per unit area.

  i.e. $\sigma = \mathrm{d}N/\mathrm{d}A$ (33)

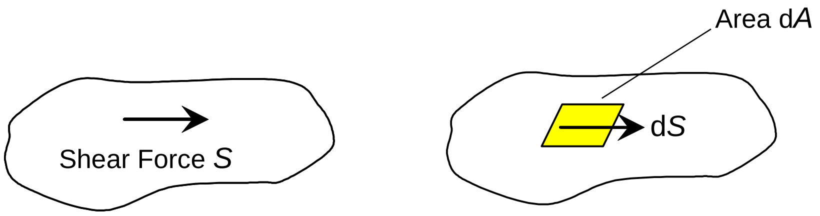
  where $\mathrm{d}A$ is differential area.
- *Shear stress* resulting from shear force - often denoted by the symbol τ - Figure 18(b). Shear stress is shear force per unit area.

  i.e. $\tau = \mathrm{d}S/\mathrm{d}A$ (34)

Stress is an internal force intensity i.e. one has to consider a ‘cut’ to interpret it.
Pressure is another form of force intensity which acts between surfaces which are in contact or within a fluid.

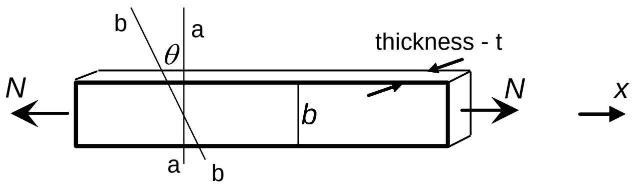


(a) Normal force and normal stress

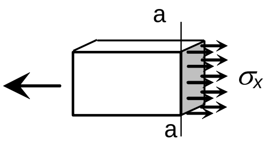


(a) Shear force and shear stress

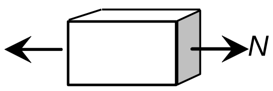
Figure 18 Definitions of normal and shear forces and stresses



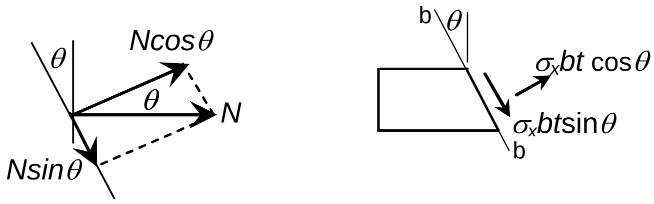
(a) Bar in tension



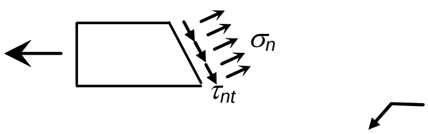
(b) Uniform stress on Section a-a



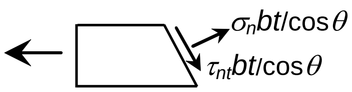
(c) Force on Section a-a



(d) Forces on Section b-b due σ_x



(e) Stress on Section b-b



(f) Forces on Section b-b due σ_n and τ_{nt}

Figure 19 Uniaxial normal stress in a bar

11.2 Uniaxial stress

In this section the distribution of stresses in the state of uniaxial tension is demonstrated.

Figure 19(a) shows a bar of rectangular cross section in uniaxial tension.

- *Axial* means that the resultant of the normal force is at the centre of area, and at right angles to, the section being considered. In this situation the standard assumption is that *the stress over the area of the bar is assumed to be constant.*
- *Tension* means that the resultant force is acting away from the section.
- *Uniaxial* means that there is no normal force/stress at right angles to the section being considered

The bar has width b, thickness t and has an applied axial load N causing a normal stress - σ_x - at the section a-a - Figure 19(b).

The relationship between the force and the stress assuming the stress to be constant is:

$$\sigma_x = N/bt \tag{35}$$

Therefore the force in the bar (Figure 19(c)) is

$$N = \sigma_x bt \tag{36}$$

At a section - bb - oriented at an angle θ to the plane (Figure 19(d)) on which σ_x acts, there will be a normal stress σ_n and a shear stress τ_n (Figure 19(e)) with values:

$$\sigma_n = \sigma_x \cos^2\theta \tag{37}$$

$$\tau_n = \sigma_x \cos\theta \sin\theta \tag{38}$$

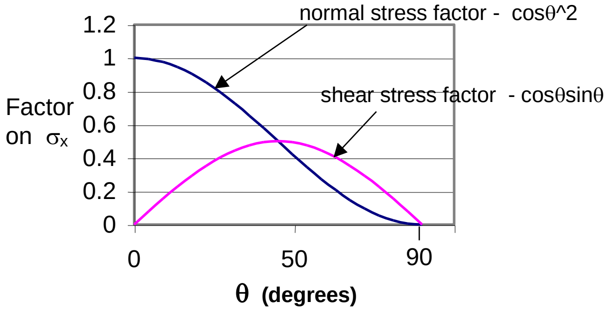


Figure 20 Variation of direct and shear stress on an inclined cross-section

Figure 20 shows the variation of direct stress and shear stress as a function of the angle of the inclined section b-b. Note that:

- The normal stress varies from σ_x at $\theta = 0.0$ to zero at $\theta = 90^o$
- The shear stress is zero at $\theta = 0.0$ and at $\theta = 90^o$ and has a maximum value of $\sigma_x/2$ at $\theta = 45^o$

Derivation of Equations (37) and (17.38)

Resolving N into components normal to, and parallel to, the inclined plane gives Figure 19(d):

Force normal to the inclined plane - $N\cos\theta = \sigma_x bt\cos\theta$ (39)

Force parallel to the plane - $N\sin\theta = \sigma_x bt\sin\theta$ (40)

The direct stress on Section b-b is defined as σ_n and the shear stress is τ_{nt} - Figure 19(e).

The area of the inclined plane is $bt/\cos\theta$ - (Figure 19(e)

Therefore the normal stress on the inclined plane is force/area i.e.:

$$\sigma_n = \text{load/area} = \sigma_x bt\cos\theta/(bt/\cos\theta) = \sigma_x\cos^2\theta$$

Similarly:

$$\tau_n = \sigma_x bt\sin\theta/(bt/\cos\theta) = \sigma_x\cos\theta\sin\theta$$

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3. Megson T H G Structural and Stress Analysis 2nd Ed ISBN: 978-07506-6221-5 Elsevier 2005